RWTH AACHEN UNIVERSITY DEPARTMENT OF PHYSICS

Theoretical Physics III (Quantum Mechanics)

Entrance Examination for Master Students

Sep-12, 2014 9:30 – 12:00

Problem 1: Operator Theory (2+3+1+4=10) points

Let A be a hermitean, linear operator in some Hilbert space $\mathcal{H}$. Assume that A has a purely discrete, non-degenerate spectrum, and denote the eigenvalues by a_k and the eigenstates by $|a_k\rangle$, $(k = 1, 2, \ldots)$.

(a) Show that the eigenvalues are real, $a_k \in \mathbb{R}$.

(b) Show that $a_k \neq a_l$ implies that $|a_k\rangle$ is orthogonal to $|a_l\rangle$.

(c) Formulate the spectral decomposition of A and the resolution of unity.

(d) Let $|\psi\rangle$ be any normalized state. Show that $\langle\psi|A|\psi\rangle \geq a_1$, where a_1 is the smallest eigenvalue of A.

Problem 2: Harmonic Oscillator (1+4+5=10) points

Consider the 1-dimensional harmonic oscillator with the normalized eigenstates $\{|n\rangle; n \in \mathbb{N}_0\}$. Assume that, at time $t = 0$, the oscillator is in the initial state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

(a) Determine $|\psi(t)\rangle$ as a linear combination of the eigenstates.

(b) Use the lowering and rising operators, a and a^+, to calculate the expectation values $\langle X\rangle_{\psi(t)}$ and $\langle P\rangle_{\psi(t)}$, where X and P are the position and momentum operator, respectively.

Formulae:

$$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^+\right) \qquad P = i\sqrt{\frac{\hbar m\omega}{2}}\left(a - a^+\right)$$

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle$$

(c) Calculate also the variances $\langle(\Delta X)^2\rangle_{\psi(t)}$ and $\langle(\Delta P)^2\rangle_{\psi(t)}$.

Problem 3: Spin-1 Particle (1+4+4+3+3=15) points

Consider a spin-1 particle with Hamilton operator

$$H = \frac{1}{I_x}(S_x)^2 + \frac{1}{I_y}(S_y)^2 + \frac{1}{I_z}(S_z)^2$$

where $I_x, I_y, I_z > 0$ with $I_x \neq I_y, I_y \neq I_z, I_z \neq I_x$.

$$S_x = \hbar \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad S_y = \hbar \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \qquad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

describe the spin operator $S = (S_x, S_y, S_z)$ in matrix representation.

(a) Which set of commuting operators determines the basis of this matrix representation?

(b) Show that in the above matrix representation H is of the form

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{pmatrix}$$

and express the constants A, B, C in terms of I_x, I_y, I_z.

(c) Determine the energy spectrum of H and the corresponding eigenvectors as functions of A, B, C.

(d) Calculate the time-evolution operator $U(t)$ in matrix representation.

(e) Consider the initial state

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and calculate the expectation value $\langle S_z \rangle_{\psi(t)} = \langle \psi(t) | S_z | \psi(t) \rangle$ as a function of time.

Problem 4: One-dimensional potential problem (1+5+2+2+1+6+3=20) points

Consider a particle of mass m in the potential

$$V(x) = -v\,\delta(x)$$

where $v > 0$ is a constant. The stationary Schrödinger equation reads

$$H\,\psi(x) \equiv \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\,\psi(x)$$

(a) What are the boundary conditions of $\psi(x)$ at $x \to \pm\infty$ for discrete eigenstates?

(b) Find all solutions of the Schrödinger equation that are everywhere continuous and satisfy the boundary conditions for discrete eigenstates.
Hint: Consider the domains $x < 0$ and $x > 0$ separately.

(c) Derive the matching condition

$$\lim_{\varepsilon\to 0+}\left[\psi'(\varepsilon) - \psi'(-\varepsilon)\right] = -\frac{2m}{\hbar^2}\,v\,\psi(0)$$

(d) Using (c), show that there is exactly one discrete eigenstate and determine the associated eigenvalue E_0.

(e) Does H also have a continuous spectrum? If yes, where is it located (i.e. which E-values belong to it)?

Consider now the more complicated potential

$$V(x) = \begin{cases} \infty & \text{for } |x| > a \\ -v\,\delta(x) & \text{for } |x| \le a \end{cases}$$

(f) What are the boundary conditions at $x = \pm a$? Find all solutions of the Schrödinger equation with negative energy that are continuous and satisfy the boundary conditions.

In case you were unable to solve (f), assume that the solutions are of the form

$$\psi(x) = A \sinh\kappa\left(a - |x|\right) \qquad (\kappa > 0)$$

where $A \in \mathbb{C}$. What is the relation between κ and E?

(g) Employ the matching condition from (c) to derive the relation

$$f(\kappa) \equiv \kappa\coth\kappa a = \frac{mv}{\hbar^2}$$

How many eigenstates with negative energy are there, depending on the parameters of the problem?
Hint: plot $f(\kappa)$ as a function of κ.